Restricted flows of the Ablowitz-Ladik hierarchy and their continuous limits

1995 J. Phys. A: Math. Gen. 28 113

(http://iopscience.iop.org/0305-4470/28/1/016)

J. Phys. A: Math. Gen. 28 (1995) 113–134. Printed in the UK

Restricted flows of the Ablowitz–Ladik hierarchy and their continuous limits*

Yunbo Zeng†§ and S Rauch-Wojciechowski‡‖

† Department of Mathematics, University of Science and Technology of China, Hefei, 230026, People's Republic of China

‡ Department of Mathematics, Linköping University, S-581 83 Linköping, Sweden

Received 13 March 1994, in final form 17 October 1994

Abstract. It is studied here how the Ablowitz–Ladik (AL) hierarchy and their restricted flows are related to the AKNS hierarchy and to their restricted flows in the continuous limit. It is shown that restricted flows of the AL hierarchy yield discrete maps approximating restricted flows of the AKNS hierarchy. Integrals of motion and Lax representation for restricted flows of the AL hierarchy as well as a discrete zero-curvature representation for the AL hierarchy with sources are given.

1. Introduction

In this paper we consider restricted flows of the Ablowitz–Ladik (AL) hierarchy in order to construct discrete maps approximating restricted flows of the AKNS hierarchy [1, 2] which are known to possess the form of multiwave interaction equations.

This work is a part of a programme aimed towards systematic construction of integrable maps approximating integrable Hamiltonian systems constructed as restricted flows of soliton hierarchy [3–6]. These maps give rise to discrete algorithms for the numerical calculation of trajectories [7]. Such algorithms are known [7, 8] to be considerably better than the standard discretizations (e.g. a multistep method) or even symplectic discretizations.

By restricted flows of a soliton hierarchy we means sets of ordinary differential equations (ODE) invariant with respect to the action of all flows of this hierarchy which are constructed in the following way: they consist of a fixed number of copies of the spectral problem and of a restriction for a (higher) flow of the hierarchy in terms of square eigenfunctions. It has been shown [1, 2, 9, 10] that, in many instances, these equations have the form of Newton equations or of a dynamical system and, therefore, can model physically interesting processes. Restricted flows of the AKNS hierarchy [1, 2] are physically interesting equations because they acquire the form of multiwave interaction systems [11] which model the growth of a low-frequency internal ocean wave by interaction with a spectrum of higher frequency waves [12] and have also been used as a model for plasma turbulence [13].

The approach for constructing restricted flows of soliton hierarchy can also be applied to obtain restricted flows (discrete maps) of a hierarchy of integrable discrete systems (nonlinear differential difference equations). We suppose that the hierarchy of integrable

* Partially supported by the Swedish Institute stipend GH/MLH ref 304/81.

§ Partially supported by Chinese National Basic Research Project.

‖ Partially supported by the NRF grant F-FU 08677.

discrete systems is associated with a discrete isospectral problem and possesses Hamiltonian structure. Then we consider a system consisting of N copies of the spectral problem and of constraint relating the variational derivatives of Hamiltonian functions and eigenvalues. This system is also invariant under all flows in the hierarchy and naturally gives rise to a discrete Euler–Lagrange equations. Integrals of motion and Lax representation for these Lagrange systems can be deduced directly from the stationary discrete zero-curvature equation for the hierarchy. In many cases, these discrete maps are discrete versions of restricted flows of some soliton hierarchy. As we shall show, the restricted flows of the AL hierarchy are discrete versions of restricted flows of the AKNS hierarchy.

We note that discrete versions of several classical integrable systems are investigated in [14]. To describe such a discrete system, a variational principle is taken as a starting point, and the Lax representation for the discrete integrable system is found via a factorization of certain matrix polynomials in [14]. It is easy to find that this approach and the approach mentioned above are quite different and are used to treat different systems. We would like to emphasize that the starting point for reducing discrete maps in our approach is a hierarchy of integrable discrete systems with Lax representation and Hamiltonian structure, and the property of these maps, such as the Lax representation, is directly deduced from one of the hierarchies.

This paper is organised as follows. In the next section, we describe the zero-curvature representation for the AL hierarchy [15], by using the method of [16], and present recursion formulae and the Hamiltonian structure for the AL hierarchy in a somewhat different way than in [15, 17, 18]. Then, in section 3, we show, by using the method in [6], how the Hamiltonian structure, the recursion relation and the square eigenfunction relation for the AL hierarchy converge to those for the AKNS hierarchy. In particular, we construct a sequence of equations in the AL hierarchy which has the AKNS hierarchy as a continuous limit. Finally, in section 4, we study restricted flows of the AL hierarchy and find integrals of motion and Lax representations for these Lagrange systems as well as the discrete zero-curvature representation for the AL hierarchy with sources. We also prove that restricted flows of the AL hierarchy go to restricted flows of the AKNS hierarchy.

2. The Hamiltonian structure of the AL hierarchy

Consider the following Ablowitz–Ladik discrete isospectral problem [15]:

$$E\psi = U\psi \qquad U = U(u, z) = \begin{pmatrix} z & Q \\ R & 1/z \end{pmatrix} \qquad \psi = (\psi_1, \psi_2)^t \tag{2.1}$$

where $u = (u_1, u_2)^t = (Q, R)^t$, $Q = Q(n, t)$ and $R = R(n, t)$ depend on integers $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, z is the spectral parameter and the shift operator E and difference operator D are defined as

$$(Ef)(n) = f(n+1) \qquad (Df)(n) = (E-1)f(n) \qquad n \in \mathbb{Z}. \tag{2.2}$$

Throughout this paper, we write $f^{(k)} = E^{(k)}f$. Some recursion formulae for a hierarchy of discrete integrable systems associated with (2.1) (referred to as an AL hierarchy) were given in [15, 17, 18]. To find the continuous limit and restricted flows of the AL hierarchy, we need their Hamiltonian structure and zero-curvature representation. So, first we present the discrete zero-curvature and Hamiltonian structures as well as the corresponding recursion formulae for the AL hierarchy. These formulae are somewhat different from those in [15, 17, 18].

We combine (2.1) with its t-evolution part

$$\psi_{t_m} = V_m \psi \qquad m = 1, 2, \ldots \tag{2.3}$$

The compatibility condition of (2.1) and (2.3) gives rise to a discrete zero-curvature equation (assuming $z_{t_m} = 0$)

$$U_{t_m} = (EV_m)U - UV_m \qquad m = 1, 2, \ldots. \tag{2.4}$$

To derive the hierarchy of evolution equations associated with (2.1), we first solve the stationary discrete zero-curvature equation [16]

$$(E\Gamma)U - U\Gamma = 0. \tag{2.5}$$

The substitution of

$$\Gamma = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{2.6}$$

into (2.5) gives

$$A^{(1)}z + B^{(1)}R - Az - CQ = 0 \tag{2.7a}$$

$$A^{(1)}Q + B^{(1)}\frac{1}{z} - Bz + AQ = 0 \tag{2.7b}$$

$$C^{(1)}z - A^{(1)}R - AR - C\frac{1}{z} = 0 \tag{2.7c}$$

$$C^{(1)}Q - A^{(1)}\frac{1}{z} - BR + A\frac{1}{z} = 0. \tag{2.7d}$$

We shall find two power-series solutions to (2.7) in powers of z and $\frac{1}{z}$, respectively.

2.1. *The AL hierarchy corresponding to Γ expanded in power series of $1/z$*

Let us assume

$$\Gamma = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} A_{2i}z^{-2i} & B_{2i+1}z^{-2i-1} \\ C_{2i+1}z^{-2i-1} & -A_{2i}z^{-2i} \end{pmatrix}. \tag{2.8}$$

Then, (2.7) leads to the following recursion relations:

$$A_0^{(1)} - A_0 = 0 \qquad B_1 = Q(A_0^{(1)} + A_0) \qquad C_1^{(1)} = R(A_0^{(1)} + A_0) \tag{2.9a}$$

$$A_{2i}^{(1)} - A_{2i} = QC_{2i-1} - RB_{2i-1}^{(1)} = QC_{2i+1}^{(1)} - RB_{2i+1} \tag{2.9b}$$

$$B_{2i+1} = Q(A_{2i}^{(1)} + A_{2i}) + B_{2i-1}^{(1)} \tag{2.9c}$$

$$C_{2i+1}^{(1)} = R(A_{2i}^{(1)} + A_{2i}) + C_{2i-1} \qquad i = 1, 2, \ldots. \tag{2.9d}$$

As the initial value for this recursion, we take

$$A_0 = \tfrac{1}{2}. \tag{2.10a}$$

Then we obtain .

$$B_1 = Q \qquad C_1^{(1)} = R \qquad A_2 = -QR^{(-1)} \tag{2.10b}$$

$$B_3 = Q^{(1)} - QQ^{(1)}R - Q^2R^{(-1)} \qquad C_3^{(1)} = R^{(-1)} - RR^{(-1)}Q - R^2Q^{(1)} \dots \tag{2.10c}$$

Let us denote

$$(\Gamma z^{2m})_+ = \begin{pmatrix} \sum_{i=0}^{m} A_{2i} z^{2m-2i} & \sum_{i=0}^{m-1} B_{2i+1} z^{2m-2i-1} \\ \sum_{i=0}^{m-1} C_{2i+1} z^{2m-2i-1} & -\sum_{i=0}^{m} A_{2i} z^{2m-2i} \end{pmatrix}. \tag{2.11}$$

It is easy to verify that

$$(E(\Gamma z^{2m})_+)U - U(\Gamma z^{2m})_+ = \begin{pmatrix} 0 & B_{2m+1} \\ -C_{2m+1}^{(1)} & (A_{2m} - A_{2m}^{(1)})\frac{1}{z} \end{pmatrix} \tag{2.12}$$

which is not compatible with U_{t_m}. Therefore, we set

$$V_m = (\Gamma z^{2m})_+ + \Delta_m \tag{2.13}$$

and try to find Δ_m such that $(EV_m)U - UV_m$ is of the following form compatible with U_{t_m}:

$$(EV_m)U - UV_m = \begin{pmatrix} 0 & g_m \\ f_m & 0 \end{pmatrix}. \tag{2.14}$$

It is easy to find that

$$\Delta_m = \begin{pmatrix} 0 & 0 \\ 0 & A_{2m} \end{pmatrix} \tag{2.15}$$

and

$$g_m = B_{2m+1} - A_{2m}Q \qquad f_m = -C_{2m+1}^{(1)} + A_{2m}^{(1)}R. \tag{2.16}$$

Then, (2.4) gives rise to the following hierarchy of equations:

$$Q_{t_m} = B_{2m+1} - A_{2m}Q \qquad R_{t_m} = -C_{2m+1}^{(1)} + A_{2m}^{(1)}R \qquad m = 1, 2, \dots. \tag{2.17}$$

In order to write hierarchy (2.17) in Hamiltonian form, we apply the following trace identity given in [16]:

$$\frac{\delta}{\delta u_i}\mathrm{Tr}\left(V\frac{\partial U}{\partial z}\right) = \left(z^{-\gamma}\left(\frac{\partial}{\partial z}\right)z^{\gamma}\right)\mathrm{Tr}\left(V\frac{\partial U}{\partial u_i}\right) \tag{2.18}$$

where Tr means trace of a matrix, γ is a constant to be fixed and V is given by

$$V = \Gamma U^{-1} = \frac{1}{1-RQ}\begin{pmatrix} A\frac{1}{z} - BR & -AQ + Bz \\ C\frac{1}{z} + AR & -CQ - Az \end{pmatrix} \tag{2.19}$$

and $\delta/\delta u_i$ stands for the discrete variational derivative defined in the usual way as

$$\frac{\delta f}{\delta u_i} = \sum_{k\in\mathbb{Z}} E^{(-k)} \frac{\partial f}{\partial u_i^{(k)}}. \tag{2.20}$$

By using (2.7), it is found that

$$\mathrm{Tr}\left(V\frac{\partial U}{\partial z}\right) = \frac{1}{1-RQ}\left(2A\frac{1}{z} - BR + CQ\frac{1}{z^2}\right) = \frac{1}{z}(A^{(1)} + A) \tag{2.21a}$$

$$\mathrm{Tr}\left(V\frac{\partial U}{\partial Q}\right) = \frac{1}{1-RQ}\left(C\frac{1}{z} + AR\right) = \frac{1}{1-RQ}(C^{(1)}z - A^{(1)}R) \tag{2.21b}$$

$$\mathrm{Tr}\left(V\frac{\partial U}{\partial R}\right) = \frac{1}{1-RQ}(Bz - AQ). \tag{2.21c}$$

Then, (2.18) gives

$$\frac{2}{z}\frac{\delta A}{\delta Q} = \left(z^{-\gamma}\left(\frac{\partial}{\partial z}\right)z^{\gamma}\right)\frac{1}{1-RQ}(C^{(1)}z - A^{(1)}R) \tag{2.22a}$$

$$\frac{2}{z}\frac{\delta A}{\delta R} = \left(z^{-\gamma}\left(\frac{\partial}{\partial z}\right)z^{\gamma}\right)\frac{1}{1-RQ}(Bz - AQ). \tag{2.22b}$$

By expanding in powers of z, we obtain

$$2\frac{\delta A_{2i}}{\delta Q} = \frac{\gamma - 2i}{1-RQ}(C^{(1)}_{2i+1} - A^{(1)}_{2i}R). \tag{2.22c}$$

To fix the constant γ, we simply set $i = 1$ and find that $\gamma = 0$. Thus, we have

$$F_i = \frac{1}{1-RQ}(C^{(1)}_{2i+1} - A^{(1)}_{2i}R) = \frac{\delta H_i}{\delta Q} \tag{2.23a}$$

$$G_i = \frac{1}{1-RQ}(B_{2i+1} - A_{2i}Q) = \frac{\delta H_i}{\delta R} \tag{2.23b}$$

where

$$H_i = -\frac{A_{2i}}{i} \qquad i = 1, 2, \ldots. \tag{2.23c}$$

For $i = 0$, we define

$$F_0 = \frac{R}{1-RQ} = \frac{\delta H_0}{\delta Q} \qquad G_0 = \frac{Q}{1-RQ} = \frac{\delta H_0}{\delta R} \qquad H_0 = -\ln(1-RQ). \tag{2.23d}$$

The recursion formula for F_i and G_i can be found in the following way. From (2.9) we have that

$$QF_i - RG_i = \frac{1}{1-RQ}(QC^{(1)}_{2i+1} - A^{(1)}_{2i}RQ - RB_{2i+1} + A_{2i}RQ) = A^{(1)}_{2i} - A_{2i}$$

and, therefore,

$$A_{2i} = D^{-1}(QF_i - RG_i) \tag{2.24}$$

where D^{-1} is the inverse of the difference operator D. By using (2.9*b*), (2.9*c*) and (2.24), it is found that

$$\begin{aligned}
F_i &= \frac{1}{1-RQ}(A_{2i}R + C_{2i-1}) \\
&= \frac{1}{1-RQ}\{RD^{-1}(QC_{2i-1} - RB^{(1)}_{2i-1}) + C_{2i-1}\} \\
&= \frac{1}{1-RQ}\{RD^{-1}[Q(C_{2i-1} - R^{(-1)}A_{2i-2}) - R(B^{(1)}_{2i-1} - Q^{(1)}A^{(1)}_{2i-2})] \\
&\quad - RQR^{(-1)}A_{2i-2} + C_{2i-1}\} \\
&= \frac{1}{1-RQ}\{RD^{-1}[QE^{(-1)}(1-RQ)F_{i-1} - RE(1-RQ)G_{i-1}] \\
&\quad + (1-RQ)R^{(-1)}D^{-1}(QF_{i-1} - RG_{i-1}) + E^{(-1)}(1-RQ)F_{i-1}\}.
\end{aligned}$$

We can also find a similar formula for G_i. We, therefore, have

$$\begin{pmatrix} F_i \\ G_i \end{pmatrix} = L\begin{pmatrix} F_{i-1} \\ G_{i-1} \end{pmatrix} = L^i\begin{pmatrix} F_0 \\ G_0 \end{pmatrix} = L^i\begin{pmatrix} R/(1-RQ) \\ Q/(1-RQ) \end{pmatrix} \qquad i = 1, 2, \ldots \tag{2.25a}$$

where

$$L = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} \tag{2.25b}$$

$$\begin{aligned}
I_{11} &= \frac{1}{1-RQ}E^{(-1)}(1-RQ) + \frac{R}{1-RQ}D^{-1}QE^{(-1)}(1-RQ) + R^{(-1)}D^{-1}Q \\
I_{12} &= -\frac{R}{1-RQ}D^{-1}RE(1-RQ) - R^{(-1)}D^{-1}R \\
I_{21} &= \frac{Q}{1-RQ}D^{-1}Q^{(1)}(1-RQ) + Q^{(1)}D^{-1}EQ \\
I_{22} &= \frac{1}{1-RQ}E(1-RQ) - \frac{Q}{1-RQ}D^{-1}R^{(1)}E^{(2)}(1-RQ) - Q^{(1)}D^{-1}ER.
\end{aligned}$$

Therefore, we can write the hierarchy (2.17) in the following Hamiltonian form

$$\begin{aligned}
\begin{pmatrix} Q \\ R \end{pmatrix}_{t_m} &= J\begin{pmatrix} F_m \\ G_m \end{pmatrix} = JL^m\begin{pmatrix} F_0 \\ G_0 \end{pmatrix} = JL^m\begin{pmatrix} R/(1-RQ) \\ Q/(1-RQ) \end{pmatrix} \\
&= J\frac{\delta H_m}{\delta u} \qquad m = 1, 2, \ldots
\end{aligned} \tag{2.26}$$

where the Hamiltonian operator J is given by

$$J = \begin{pmatrix} 0 & 1-RQ \\ -1+RQ & 0 \end{pmatrix} \qquad H_m = -\frac{A_{2m}}{m} \qquad m = 1, 2, \ldots. \tag{2.27}$$

2.2. The AL hierarchy corresponding to Γ expanded in power series of z

Similarly, if we take Γ in (2.6) as follows

$$\bar{\Gamma} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & -\bar{A} \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} \bar{A}_{2i} z^{2i} & \bar{B}_{2i+1} z^{2i+1} \\ \bar{C}_{2i+1} z^{2i+1} & -\bar{A}_{2i} z^{2i} \end{pmatrix} \tag{2.28}$$

then (2.7) leads to the following recursion relations:

$$\bar{A}_0^{(1)} - \bar{A}_0 = 0 \qquad \bar{B}_1^{(1)} = -Q(\bar{A}_0^{(1)} + \bar{A}_0) \qquad \bar{C}_1 = -R(\bar{A}_0^{(1)} + \bar{A}_0) \tag{2.29a}$$

$$\bar{A}_{2i}^{(1)} - \bar{A}_{2i} = Q\bar{C}_{2i-1}^{(1)} - R\bar{B}_{2i-1} = Q\bar{C}_{2i+1} - R\bar{B}_{2i+1}^{(1)} \tag{2.29b}$$

$$\bar{B}_{2i+1}^{(1)} = -Q(\bar{A}_{2i}^{(1)} + \bar{A}_{2i}) + \bar{B}_{2i-1} \tag{2.29c}$$

$$\bar{C}_{2i+1} = -R(\bar{A}_{2i}^{(1)} + \bar{A}_{2i}) + \bar{C}_{2i-1}^{(1)} \qquad i = 1, 2, \ldots. \tag{2.29d}$$

As the initial value for this recursion we take

$$\bar{A}_0 = \tfrac{1}{2}. \tag{2.30a}$$

Then, we obtain

$$\bar{B}_1^{(1)} = -Q \qquad \bar{C}_1 = -R \qquad \bar{A}_2 = -RQ^{(-1)} \tag{2.30b}$$

$$\bar{B}_3^{(1)} = -Q^{(-1)} + QQ^{(-1)}R + Q^2R^{(1)} \qquad \bar{C}_3 = -R^{(1)} + RR^{(1)}Q + R^2Q^{(-1)} \tag{2.30c}$$

Furthermore, in the same way as for (2.11)–(2.15), we obtain

$$\begin{aligned} \bar{V}_m &= (\bar{\Gamma} z^{-2m})_- + \bar{\Delta}_m \\ &= \begin{pmatrix} \sum_{i=0}^{m} \bar{A}_{2i} z^{-2m+2i} & \sum_{i=0}^{m-1} \bar{B}_{2i+1} z^{-2m+2i+1} \\ \sum_{i=0}^{m-1} \bar{C}_{2i+1} z^{-2m+2i+1} & -\sum_{i=0}^{m} \bar{A}_{2i} z^{-2m+2i} \end{pmatrix} + \begin{pmatrix} -\bar{A}_{2m} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.31}$$

Now we substitute V_m in (2.3) with $\bar{V}_m$, then (2.4), with $V_m = \bar{V}_m$, gives rise to the following hierarchy of equations:

$$Q_{t_m} = -\bar{B}_{2m+1}^{(1)} - \bar{A}_{2m}^{(1)} Q \qquad R_{t_m} = \bar{C}_{2m+1} + \bar{A}_{2m} R. \tag{2.32}$$

Then, we find that

$$\bar{F}_i = -\frac{1}{1 - RQ}(\bar{C}_{2i+1} + \bar{A}_{2i} R) = \frac{\delta \bar{H}_i}{\delta Q} \tag{2.33a}$$

$$\bar{G}_i = -\frac{1}{1 - RQ}(\bar{B}_{2i+1}^{(1)} + \bar{A}_{2i}^{(1)} Q) = \frac{\delta \bar{H}_i}{\delta R} \qquad i = 1, 2, \ldots, \tag{2.33b}$$

where

$$\bar{H}_i = -\frac{\bar{A}_{2i}}{i} \qquad i = 1, 2, \ldots. \tag{2.33c}$$

For $i = 0$, we define

$$\bar{F}_0 = \frac{R}{1-RQ} = \frac{\delta \bar{H}_0}{\delta Q} \qquad \bar{G}_0 = \frac{Q}{1-RQ} = \frac{\delta \bar{H}_0}{\delta R} \qquad \bar{H}_0 = -\ln(1-RQ). \tag{2.33d}$$

We also have

$$\begin{pmatrix} \bar{F}_i \\ \bar{G}_i \end{pmatrix} = \bar{L} \begin{pmatrix} \bar{F}_{i-1} \\ \bar{G}_{i-1} \end{pmatrix} = \bar{L}^i \begin{pmatrix} \bar{F}_0 \\ \bar{G}_0 \end{pmatrix} = \bar{L}^i \begin{pmatrix} R/(1-RQ) \\ Q/(1-RQ) \end{pmatrix} \qquad i = 1, 2, \ldots \tag{2.34a}$$

where

$$\bar{L} = \begin{pmatrix} \bar{l}_{11} & \bar{l}_{12} \\ \bar{l}_{21} & \bar{l}_{22} \end{pmatrix} \tag{2.34b}$$

$$\bar{l}_{11} = \frac{1}{1-RQ} E(1-RQ) - \frac{R}{1-RQ} D^{-1} Q^{(1)} E^{(2)} (1-RQ) - R^{(1)} D^{-1} E Q$$

$$\bar{l}_{12} = \frac{R}{1-RQ} D^{-1} R^{(1)} (1-RQ) + R^{(1)} D^{-1} E R$$

$$\bar{l}_{21} = -\frac{Q}{1-RQ} D^{-1} Q E (1-RQ) - Q^{(-1)} D^{-1} Q$$

$$\bar{l}_{22} = \frac{1}{1-RQ} E^{(-1)} (1-RQ) + \frac{Q}{1-RQ} D^{-1} R E^{(-1)} (1-RQ) + Q^{(-1)} D^{-1} R.$$

Therefore, we can write hierarchy (2.32) in the following Hamiltonian form:

$$\begin{pmatrix} Q \\ R \end{pmatrix}_{t_m} = J \begin{pmatrix} \bar{F}_m \\ \bar{G}_m \end{pmatrix} = J \bar{L}^m \begin{pmatrix} \bar{F}_0 \\ \bar{G}_0 \end{pmatrix} = J \bar{L}^m \begin{pmatrix} R/(1-RQ) \\ Q/(1-RQ) \end{pmatrix} = J \frac{\delta \bar{H}_m}{\delta u} \qquad m = 1, 2 \ldots \tag{2.35a}$$

where the Hamiltonian operator J is given by (2.27) and

$$\bar{H}_m = -\frac{\bar{A}_{2m}}{m} \qquad m = 1, 2, \ldots. \tag{2.35b}$$

Finally, by combining (2.26) and (2.35), we obtain the following proposition.

Proposition 1. If we take

$$\psi_{t_m} = (\alpha_1 V_m + \alpha_2 \bar{V}_m) \psi \tag{2.36}$$

where α_1 and α_2 are arbitrary constants, V_m and $\bar{V}_m$ are given by (2.13) and (2.31), respectively, and the discrete hierarchy of zero-curvature equations (2.4) then gives rise to the following AL hierarchy:

$$\begin{aligned} \begin{pmatrix} Q \\ R \end{pmatrix}_{t_m} &= \alpha_1 J \begin{pmatrix} F_m \\ G_m \end{pmatrix} + \alpha_2 J \begin{pmatrix} \bar{F}_m \\ \bar{G}_m \end{pmatrix} \\ &= \alpha_1 J L^m \begin{pmatrix} F_0 \\ G_0 \end{pmatrix} + \alpha_2 J \bar{L}^m \begin{pmatrix} \bar{F}_0 \\ \bar{G}_0 \end{pmatrix} \\ &= J \frac{\delta(\alpha_1 H_m + \alpha_2 \bar{H}_m)}{\delta u} \qquad m = 1, 2, \ldots \end{aligned} \tag{2.37}$$

where the Hamiltonian operator J is given by (2.27), recursion operator L and $\bar{L}$ by (2.25*b*) and (2.34*b*), $(F_0, G_0)^t$ and $(\bar{F}_0, \bar{G}_0)^t$ by (2.23*d*) and (2.33*d*) and H_m and $\bar{H}_m$ by (2.27) and (2.35*b*).

The adjoint equation for (2.1) reads:

$$E^{(-1)}\phi = \phi U \qquad \phi = (\phi_1, \phi_2) \tag{2.38}$$

and it can be found by a direct calculation that the square eigenfunctions

$$\frac{\delta z}{\delta Q} = \psi_2\phi_1 \qquad \text{and} \qquad \frac{\delta z}{\delta R} = \psi_1\phi_2 \tag{2.39}$$

satisfy

$$L\frac{\delta z}{\delta u} = L\begin{pmatrix} \psi_2\phi_1 \\ \psi_1\phi_2 \end{pmatrix} = z^2\frac{\delta z}{\delta u} = z^2\begin{pmatrix} \psi_2\phi_1 \\ \psi_1\phi_2 \end{pmatrix} \tag{2.40a}$$

$$\bar{L}\frac{\delta z}{\delta u} = \bar{L}\begin{pmatrix} \psi_2\phi_1 \\ \psi_1\phi_2 \end{pmatrix} = \frac{1}{z^2}\frac{\delta z}{\delta u} = \frac{1}{z^2}\begin{pmatrix} \psi_2\phi_1 \\ \psi_1\phi_2 \end{pmatrix}. \tag{2.40b}$$

3. The continuous limit of the AL hierarchy

It is known that some nonlinear differential-difference equations in the AL hierarchy are discrete versions of some soliton equations, such as discrete mKdV [15]. We shall show here that the Hamiltonian structure, the recursion relation and the square eigenfunction relation for the AL hierarchy converge to those for the AKNS hierarchy, and the continuous limit of a sequence of equations in the AL hierarchy (2.37) yields the AKNS hierarchy.

The AKNS spectral problem is [19]

$$\tilde{\psi}_x = M\tilde{\psi} \qquad M = M(\tilde{u}, \lambda) = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix} \qquad \tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2)^t \tag{3.1}$$

where $\tilde{u} = (q, r)^t$ and λ is the spectral parameter. The AKNS hierarchy associated with (3.1) reads:

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_m} = \rho J_0 \begin{pmatrix} c_{m+2} \\ b_{m+2} \end{pmatrix} = \rho J_0 L_0^{m+1} \begin{pmatrix} r \\ q \end{pmatrix} = \rho J_0 \frac{\delta \tilde{H}_{m+2}}{\delta \tilde{u}} \qquad m = 1, 2, \ldots \tag{3.2}$$

where ρ is a constant and

$$\begin{pmatrix} c_{k+1} \\ b_{k+1} \end{pmatrix} = L_0 \begin{pmatrix} c_k \\ b_k \end{pmatrix} = L_0^k \begin{pmatrix} r \\ q \end{pmatrix} \tag{3.3a}$$

$$a_k = \partial^{-1}(qc_k - rb_k) \qquad \tilde{H}_m = \frac{2}{m+1}a_{m+1} \tag{3.3b}$$

$$J_0 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \qquad L_0 = \frac{1}{2}\begin{pmatrix} \partial - 2r\partial^{-1}q & 2r\partial^{-1}r \\ -2q\partial^{-1}q & -\partial + 2q\partial^{-1}r \end{pmatrix}. \tag{3.3c}$$

Here, $\partial = \partial/\partial x$ and $\partial\partial^{-1} = \partial^{-1}\partial = 1$. In particular, we have

$$\begin{aligned} &a_0 = -1 \qquad b_0 = c_0 = a_1 = 0 \qquad c_1 = r \\ &b_1 = q \qquad a_2 = \tfrac{1}{2}qr \qquad c_2 = \tfrac{1}{2}r_x \\ &b_2 = -\tfrac{1}{2}q_x \qquad c_3 = \tfrac{1}{4}(r_{xx} - 2qr^2) \qquad b_3 = \tfrac{1}{4}(q_{xx} - 2q^2r), \ldots \end{aligned} \tag{3.4}$$

If we take $r = \mp q^*$ and $\rho = 2i$, then (3.2), for $m = 1$, gives the nonlinear Schrödinger (NLS) equation

$$iq_t = q_{xx} \pm 2q^2q^*. \tag{3.5}$$

The adjoint equation for (3.1) reads:

$$\tilde{\phi}_x = -\tilde{\phi}M \qquad \tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2). \tag{3.6}$$

It is known that the variational derivatives of eigenvalues have the following expression in terms of eigenfunctions:

$$\frac{\delta\lambda}{\delta\tilde{u}} = \begin{pmatrix} \dfrac{\delta\lambda}{\delta q} \\ \dfrac{\delta\lambda}{\delta r} \end{pmatrix} = \begin{pmatrix} \tilde{\psi}_2\tilde{\phi}_1 \\ \tilde{\psi}_1\tilde{\phi}_2 \end{pmatrix} \tag{3.7}$$

and that

$$L_0\frac{\delta\lambda}{\delta\tilde{u}} = L_0\begin{pmatrix} \tilde{\psi}_2\tilde{\phi}_1 \\ \tilde{\psi}_1\tilde{\phi}_2 \end{pmatrix} = \lambda\frac{\delta\lambda}{\delta\tilde{u}} = \lambda\begin{pmatrix} \tilde{\psi}_2\tilde{\phi}_1 \\ \tilde{\psi}_1\tilde{\phi}_2 \end{pmatrix}. \tag{3.8}$$

Let us consider the AL hierarchy on a lattice with a small step h. It is known [15] that if we take

$$E^{(k)}Q(n,t) = hq(x+kh,t) \qquad E^{(k)}R(n,t) = hr(x+kh,t) \tag{3.9a}$$

$$E^{(k)}\psi_i(n,t) = \beta\tilde{\psi}_i(x+kh,t) \qquad E^{(k)}\phi_i(n,t) = \beta\tilde{\phi}_i(x+kh,t) \qquad i = 1,2 \tag{3.9b}$$

$$z = e^{-h\lambda} \tag{3.9c}$$

where β is a constant, then we have

$$E\psi_1 - z\psi_1 - Q\psi_2 = \beta h(\tilde{\psi}_{1x} + \lambda\tilde{\psi}_1 - q\tilde{\psi}_2) + O(h^2) \tag{3.10a}$$

$$E\psi_2 - R\psi_1 - \frac{1}{z}\psi_2 = \beta h(\tilde{\psi}_{2x} - \lambda\tilde{\psi}_2 - r\tilde{\psi}_1) + O(h^2) \tag{3.10b}$$

$$E^{(-1)}\phi_1 - z\phi_1 - R\phi_2 = -\beta h(\tilde{\phi}_{1x} - \lambda\tilde{\phi}_1 + r\tilde{\phi}_2) + O(h^2) \tag{3.10c}$$

$$E^{(-1)}\phi_2 - Q\phi_1 - \frac{1}{z}\phi_2 = -\beta h(\tilde{\phi}_{2x} + \lambda\tilde{\phi}_2 + q\tilde{\phi}_1) + O(h^2) \tag{3.10d}$$

which implies that (2.1) and (2.38) go to (3.1) and (3.6), respectively, in the continuous limit.

Notice that

$$E^{(k)} = e^{kh\partial} = 1 + kh\partial + O(h^2) \tag{3.11a}$$

$$D = h\partial + \tfrac{1}{2}h^2\partial^2 + O(h^3) \qquad D^{-1} = h^{-1}\partial^{-1} - \tfrac{1}{2} + O(h). \tag{3.11b}$$

Owing to (3.9), it is found that the Hamiltonian operator J given by (2.27) has the following expansion:

$$J = -\tfrac{1}{2}J_0 - \tfrac{1}{2}qrJ_0h^2 + O(h^3). \tag{3.12}$$

Similarly, for the recursion operator (2.25*b*) and (2.34*b*), we find, from (3.9) and (3.11), that

$$L = I - 2hL_0 + h^2L_1 + O(h^3) \tag{3.13a}$$

$$\bar{L} = I + 2hL_0 + h^2L_1 + O(h^3) \tag{3.13b}$$

where

$$\begin{aligned} L_1 &= \begin{pmatrix} \frac{1}{2}\partial^2 - r\partial^{-1}q\partial - rq - r_x\partial^{-1}q & -r\partial^{-1}r\partial + r^2 + r_x\partial^{-1}r \\ -q\partial^{-1}q\partial + q^2 + q_x\partial^{-1}q & \frac{1}{2}\partial^2 - q\partial^{-1}r\partial - rq - q_x\partial^{-1}r \end{pmatrix} \\ &= L_0\begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix} + \begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix}\begin{pmatrix} -r\partial^{-1}q & r\partial^{-1}r \\ -q\partial^{-1}q & q\partial^{-1}r \end{pmatrix} \\ &= L_0(2L_0 - I_0) + \tfrac{1}{2}(2L_0 - I_0)I_0 \\ &= 2L_0^2 - \tfrac{1}{2}I_0^2 \end{aligned} \tag{3.14a}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_0 = \begin{pmatrix} -2r\partial^{-1}q & 2r\partial^{-1}r \\ -2q\partial^{-1}q & 2q\partial^{-1}r \end{pmatrix}. \tag{3.14b}$$

It is easy to see that, owing to (3.9),

$$\psi_2\phi_1 = \beta^2\tilde{\psi}_2\tilde{\phi}_1 \qquad \psi_1\phi_2 = \beta^2\tilde{\psi}_1\tilde{\phi}_2 \tag{3.15}$$

and

$$\begin{aligned} L - z^2I &= I - 2hL_0 - (1 - 2h\lambda)I + O(h^2) \\ &= -2h(L_0 - \lambda I) + O(h^2) \end{aligned} \tag{3.16}$$

and, therefore, we obtain

$$(L - z^2)\begin{pmatrix} \psi_2\phi_1 \\ \psi_1\phi_2 \end{pmatrix} = -2\beta^2h(L_0 - \lambda I)\begin{pmatrix} \tilde{\psi}_2\tilde{\phi}_1 \\ \tilde{\psi}_1\tilde{\phi}_2 \end{pmatrix} + O(h^2) \tag{3.17}$$

which implies that (2.40*a*) goes to (3.8) in the continuous limit. It is known [15, 19] that for the properly defined square eigenfunctions

$$\begin{pmatrix} \psi_2\phi_1 \\ \psi_1\phi_2 \end{pmatrix} = \sum_{i=0}^{\infty}\begin{pmatrix} F_i \\ G_i \end{pmatrix} z^{-2i} = \sum_{i=0}^{\infty}\begin{pmatrix} \bar{F}_i \\ \bar{G}_i \end{pmatrix} z^{2i} \tag{3.18a}$$

$$\begin{pmatrix} \tilde{\psi}_2\tilde{\phi}_1 \\ \tilde{\psi}_1\tilde{\phi}_2 \end{pmatrix} = \sum_{i=0}^{\infty}\begin{pmatrix} c_i \\ b_i \end{pmatrix} \lambda^{-i} \tag{3.18b}$$

and, owing to (3.17), recursion relation (2.25*a*) corresponds to recursion relation (3.3).

From (3.13*a*), we obtain

$$\begin{pmatrix} F_0 \\ G_0 \end{pmatrix} = \frac{1}{1-RQ}\begin{pmatrix} R \\ Q \end{pmatrix} = h\begin{pmatrix} r \\ q \end{pmatrix} + O(h^3)$$

$$(L-I)\begin{pmatrix} F_0 \\ G_0 \end{pmatrix} = \begin{pmatrix} F_1 \\ G_1 \end{pmatrix} - \begin{pmatrix} F_0 \\ G_0 \end{pmatrix} = (-2hL_0 + h^2L_1)h\begin{pmatrix} r \\ q \end{pmatrix} + O(h^4)$$

and in general

$$\begin{aligned}(L-I)^m\begin{pmatrix} F_0 \\ G_0 \end{pmatrix} &= \sum_{i=0}^{m} C_m^i(-1)^{m-i}\begin{pmatrix} F_i \\ G_i \end{pmatrix} \\ &= (-2hL_0 + h^2L_1)^m h\begin{pmatrix} r \\ q \end{pmatrix} + O(h^{m+3}) \\ &= (-2)^{m-1}h^{m+1}[-2L_0^m + h\{L_0^{m-1}, L_1\}]\begin{pmatrix} r \\ q \end{pmatrix} + O(h^{m+3})\end{aligned} \tag{3.19}$$

where the bracket $\{\, ,\}$ means

$$\{L_0^{m-1}, L_1\} = L_0^{m-1}L_1 + L_0^{m-2}L_1L_0 + \cdots + L_0L_1L_0^{m-2} + L_1L_0^{m-1}.$$

It is easy to show, according to definition (3.14*b*), that

$$I_0\begin{pmatrix} r \\ q \end{pmatrix} = 0 \qquad I_0^2\begin{pmatrix} f \\ g \end{pmatrix} = 0$$

for any f and g, and, therefore,

$$\{L_0^{m-1}, I_0^2\}\begin{pmatrix} r \\ q \end{pmatrix} = 0$$

and by using (3.14), we obtain

$$\{L_0^{m-1}, L_1\}\begin{pmatrix} r \\ q \end{pmatrix} = [2\{L_0^{m-1}, L_0^2\} - \tfrac{1}{2}\{L_0^{m-1}, I_0^2\}]\begin{pmatrix} r \\ q \end{pmatrix} = 2mL_0^{m+1}\begin{pmatrix} r \\ q \end{pmatrix}.$$

Then, we obtain, from (3.12) and (3.19),

$$J\sum_{i=0}^{m} C_m^i(-1)^{m-i}\begin{pmatrix} F_i \\ G_i \end{pmatrix} = -\tfrac{1}{2}J_0(-2)^{m-1}h^{m+1}[-2L_0^m + 2mhL_0^{m+1}]\begin{pmatrix} r \\ q \end{pmatrix} + O(h^{m+3}). \tag{3.20}$$

Similarly, we find

$$\left(\bar{L} - \frac{1}{z^2}\right)\begin{pmatrix} \psi_2\phi_1 \\ \psi_1\phi_2 \end{pmatrix} = 2\beta^2 h(L_0 - \lambda I)\begin{pmatrix} \tilde{\psi}_2\tilde{\phi}_1 \\ \tilde{\psi}_1\tilde{\phi}_2 \end{pmatrix} + O(h^2) \tag{3.21}$$

which means that (2.40*b*) goes to (3.8) in the continuous limit. Also, (3.21) and (3.18) imply that recursion formula (2.34*a*) corresponds to recursion formula (3.3). In the same way, we get, from (3.13*b*), that

$$J\sum_{i=0}^{m} C_m^i(-1)^{m-i}\begin{pmatrix}\bar{F}_i\\ \bar{G}_i\end{pmatrix} = -\tfrac{1}{2}J_0 2^{m-1}h^{m+1}[2L_0^m + 2mhL_0^{m+1}]\begin{pmatrix}r\\ q\end{pmatrix} + \mathrm{O}(h^{m+3}). \tag{3.22}$$

By combining (3.20) and (3.22), we obtain

$$J\sum_{i=0}^{m} C_m^i(-1)^{m-i}\left[\begin{pmatrix}F_i\\ G_i\end{pmatrix} + (-1)^{m-1}\begin{pmatrix}\bar{F}_i\\ \bar{G}_i\end{pmatrix}\right] = (-2)^m mh^{m+2}J_0L_0^{m+1}\begin{pmatrix}r\\ q\end{pmatrix} + \mathrm{O}(h^{m+3}) \tag{3.23}$$

and we can, therefore, formulate the following proposition.

Proposition 2. A sequence of equations in the AL hierarchy (2.37) relates to the AKNS hierarchy in the continuous limit in the following way:

$$\begin{pmatrix}Q\\ R\end{pmatrix}_{t_m} - \frac{\rho}{(-2)^m mh^{m+1}}J\sum_{i=0}^{m} C_m^i(-1)^{m-i}\left[\begin{pmatrix}F_i\\ G_i\end{pmatrix} + (-1)^{m-1}\begin{pmatrix}\bar{F}_i\\ \bar{G}_i\end{pmatrix}\right]$$
$$= h\left[\begin{pmatrix}q\\ r\end{pmatrix}_{t_m} - \rho J_0L_0^{m+1}\begin{pmatrix}r\\ q\end{pmatrix}\right] + \mathrm{O}(h^2). \tag{3.24}$$

Equation (3.24) implies that the following sequence of equations in the AL hierarchy:

$$\begin{pmatrix}Q\\ R\end{pmatrix}_{t_m} = \frac{\rho}{(-2)^m mh^{m+1}}J\sum_{i=0}^{m} C_m^i(-1)^{m-i}\left[\begin{pmatrix}F_i\\ G_i\end{pmatrix} + (-1)^{m-1}\begin{pmatrix}\bar{F}_i\\ \bar{G}_i\end{pmatrix}\right] \tag{3.25}$$

goes to the AKNS hierarchy (3.2) in the continuous limit. For example, for $m = 1$, (3.24) becomes

$$\begin{pmatrix}Q\\ R\end{pmatrix}_{t_1} - \frac{\rho}{4h^2}J_0\begin{pmatrix}R^{(1)} + R^{(-1)} - RR^{(-1)}Q - RR^{(1)}Q - 2R\\ Q^{(1)} + Q^{(-1)} - QQ^{(-1)}R - QQ^{(1)}R - 2Q\end{pmatrix}$$
$$= h\left[\begin{pmatrix}q\\ r\end{pmatrix}_{t_1} - \frac{1}{4}\rho J_0\begin{pmatrix}r_{xx} - 2qr^2\\ q_{xx} - 2q^2r\end{pmatrix}\right] + \mathrm{O}(h^2). \tag{3.26}$$

If we take $R = \mp Q^*$ and $\rho = 2i$, the left-hand side of (3.26) gives

$$Q_{t_1} = -\frac{i}{h^2}[Q^{(1)} + Q^{(-1)} \pm QQ^*(Q^{(-1)} + Q^{(1)}) - 2Q] \tag{3.27}$$

which is the so-called differential-difference NSE. Equation (3.26) implies that (3.27) goes to the NSE (3.5) in the continuous limit.

4. Restricted flows of the AL hierarchy

We consider for N distinct z_j, $j = 1, \ldots, N$, the following system of equations consisting of replicas of (2.1) and (2.38), as well as of the restriction of variational derivatives for conserved quantities H_{k_0} and z_j:

$$E\psi_{1j} = z_j\psi_{1j} + Q\psi_{2j} \qquad E\psi_{2j} = R\psi_{1j} + \frac{1}{z_j}\psi_{2j} \qquad j = 1, \ldots, N \tag{4.1a}$$

$$E^{(-1)}\phi_{1j} = z_j\phi_{1j} + R\phi_{2j} \qquad E^{(-1)}\phi_{2j} = Q\phi_{1j} + \frac{1}{z_j}\phi_{2j} \qquad j = 1, \ldots, N \tag{4.1b}$$

$$\frac{\delta H_{k_0}}{\delta u} - \sum_{j=1}^{N} \frac{\delta z_j}{\delta u} = 0. \tag{4.1c}$$

As argued in [9, 10, 1–5], this system is invariant with respect to the action of all flows of the AL hierarchy and admits a natural Lagrangian formulation. We call (4.1) the restricted flows of the AL hierarchy. We shall show that integrals of motion and Lax representation for (4.1) can be directly derived from the stationary discrete zero-curvature equation (2.5). One of the main reasons for studying (4.1) is that they provide discrete versions of the restricted flows of the AKNS hierarchy which are finite-dimensional completely-integrable Hamiltonian systems.

We shall denote the inner product in $\mathbb{R}^N$ by $\langle \cdot, \cdot \rangle$ and use the following notation:

$$\Psi_i = (\psi_{i1}, \ldots, \psi_{iN})^t \qquad \Phi_i = (\phi_{i1}, \ldots, \phi_{iN})^t \qquad i = 1, 2 \qquad \Lambda = \mathrm{diag}(z_1, \ldots, z_N).$$

By substituting (2.39) into (4.1), we obtain

$$E\Psi_1 = \Lambda\Psi_1 + Q\Psi_2 \qquad E\Psi_2 = R\Psi_1 + \Lambda^{-1}\Psi_2 \tag{4.2a}$$

$$E^{(-1)}\Phi_1 = \Lambda\Phi_1 + R\Phi_2 \qquad E^{(-1)}\Phi_2 = Q\Phi_1 + \Lambda^{-1}\Phi_2 \tag{4.2b}$$

$$\frac{\delta H_{k_0}}{\delta Q} = \langle \Psi_2, \Phi_1 \rangle \qquad \frac{\delta H_{k_0}}{\delta R} = \langle \Psi_1, \Phi_2 \rangle \tag{4.2c}$$

which are discrete Euler–Lagrange equations [20]

$$\frac{\delta \mathcal{L}}{\delta \Phi_i} = 0 \qquad \frac{\delta \mathcal{L}}{\delta \Psi_i} = 0 \qquad i = 1, 2 \qquad \frac{\delta \mathcal{L}}{\delta R} = 0 \qquad \frac{\delta \mathcal{L}}{\delta Q} = 0 \tag{4.3a}$$

with the Lagrangian

$$\mathcal{L} = \langle \Psi_1^{(1)}, \Phi_1 \rangle + \langle \Psi_2^{(1)}, \Phi_2 \rangle - Q\langle \Psi_2, \Phi_1 \rangle - R\langle \Psi_1, \Phi_2 \rangle - \langle \Lambda\Psi_1, \Phi_1 \rangle - \langle \Lambda^{-1}\Psi_2, \Phi_2 \rangle + H_{k_0}. \tag{4.3b}$$

For example, equations (4.2), for $k_0 = 1$, read

$$E\Psi_1 = \Lambda\Psi_1 + Q\Psi_2 \qquad E\Psi_2 = R\Psi_1 + \Lambda^{-1}\Psi_2 \tag{4.4a}$$

$$E^{(-1)}\Phi_1 = \Lambda\Phi_1 + R\Phi_2 \qquad E^{(-1)}\Phi_2 = Q\Phi_1 + \Lambda^{-1}\Phi_2 \tag{4.4b}$$

$$R^{(-1)} = \langle \Psi_2, \Phi_1 \rangle \qquad Q^{(1)} = \langle \Psi_1, \Phi_2 \rangle. \tag{4.4c}$$

It is easy to see from (4.2*a*) and (4.2*b*) that

$$\begin{aligned}\langle \Lambda^i \Psi_1^{(1)}, \Phi_1\rangle + \langle \Lambda^i \Psi_2^{(1)}, \Phi_2\rangle &= \langle \Lambda^i \Psi_1, \Phi_1^{(-1)}\rangle + \langle \Lambda^i \Psi_2, \Phi_2^{(-1)}\rangle \\ &= \langle \Lambda^{i+1}\Psi_1, \Phi_1\rangle + Q\langle \Lambda^i \Psi_2, \Phi_1\rangle + R\langle \Lambda^i \Psi_1, \Phi_2\rangle + \langle \Lambda^{i-1}\Psi_2, \Phi_2\rangle\end{aligned}$$

which means that the following h_i are integrals of motion for the Lagrangian system (4.2):

$$h_i = \langle \Lambda^i \Psi_1, \Phi_1^{(-1)}\rangle + \langle \Lambda^i \Psi_2, \Phi_2^{(-1)}\rangle \qquad i = 0, 1, \ldots. \tag{4.5}$$

However, there are only N independent integrals of motion among h_i. In order to find more independent integrals of motion for (4.2), we need the following formula obtained from (2.5) and $\Gamma = VU$:

$$D\Gamma = [U, V] \tag{4.6a}$$

or

$$\Gamma^{(1)} = UV. \tag{4.6b}$$

Thus,

$$D(\operatorname{Tr}\Gamma^2) = \operatorname{Tr}\Gamma^{(1)^2} - \operatorname{Tr}\Gamma^2 = \operatorname{Tr}(UV)^2 - \operatorname{Tr}(VU)^2 = 0$$

which leads to

$$D(\operatorname{Tr}\Gamma^2) = 2D(A^2 + BC) = 0,$$

or

$$\operatorname{Tr}\Gamma^2 = 2(A^2 + BC) = \text{const}. \tag{4.7}$$

This implies that if Γ satisfies (2.5), then

$$F_k = \sum_{i=0}^{k} A_{2i}A_{2k-2i} + \sum_{i=0}^{k-1} B_{2i+1}C_{2k-2i-1} \qquad k = 0, 1, \ldots \tag{4.8}$$

are integrals of motion for the Lagrangian system (4.2). To find F_k with the simplest expression in terms of coordinates $(\Psi_1, \Psi_2, Q, \Phi_1, \Phi_2, R)$, we have to construct $\tilde{\Gamma}$ from (4.2) so that $\tilde{\Gamma}$ satisfies (2.5) and $\tilde{A}, \tilde{B}, \tilde{C}$ have the simplest expressions in terms of $(\Psi_1, \Psi_2, Q, \Phi_1, \Phi_2, R)$.

Proposition 3. Let us define

$$\tilde{A}_{2i} = A_{2i} \qquad \tilde{B}_{2i+1} = B_{2i+1} \qquad \tilde{C}_{2i+1} = C_{2i+1} \qquad i = 0, \ldots, k_0 - 1 \tag{4.9a}$$

$$\tilde{A}_{2k_0} = \langle \Psi_1, \Phi_1^{(-1)}\rangle \tag{4.9b}$$

$$\tilde{A}_{2i} = \tfrac{1}{2}(\langle \Lambda^{2(i-k_0)}\Psi_1, \Phi_1^{(-1)}\rangle - \langle \Lambda^{2(i-k_0)}\Psi_2, \Phi_2^{(-1)}\rangle) \qquad i \geqslant k_0 + 1 \tag{4.9c}$$

$$\tilde{B}_{2i+1} = \langle \Lambda^{2(i-k_0)+1}\Psi_1, \Phi_2^{(-1)}\rangle \qquad i \geqslant k_0 \tag{4.9d}$$

$$\tilde{C}_{2i+1} = \langle \Lambda^{2(i-k_0)+1}\Psi_2, \Phi_1^{(-1)}\rangle \qquad i \geqslant k_0 \tag{4.9e}$$

then $\tilde{A}_{2i}$, $\tilde{B}_{2i+1}$, $\tilde{C}_{2i+1}$, under (4.2), satisfy (2.9), namely,

$$\tilde{\Gamma} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & -\tilde{A} \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} \tilde{A}_{2i} z^{-2i} & \tilde{B}_{2i+1} z^{-2i-1} \\ \tilde{C}_{2i+1} z^{-2i-1} & -\tilde{A}_{2i} z^{-2i} \end{pmatrix}$$

under (4.2), satisfies (2.5), so we have the following integrals of motion for the Lagrangian system (4.2):

$$F_k = \sum_{i=0}^{k} \tilde{A}_{2i} \tilde{A}_{2k-2i} + \sum_{i=0}^{k-1} \tilde{B}_{2i+1} \tilde{C}_{2k-2i-1} \qquad k = 0, 1, \ldots. \tag{4.10}$$

Proof. From (4.2*c*), we obtain

$$\begin{pmatrix} F_{k_0} \\ G_{k_0} \end{pmatrix} = \begin{pmatrix} \langle \Psi_2, \Phi_1 \rangle \\ \langle \Psi_1, \Phi_2 \rangle \end{pmatrix}. \tag{4.11a}$$

According to (2.25) and (2.40*a*), let us define

$$\begin{pmatrix} \tilde{F}_i \\ \tilde{G}_i \end{pmatrix} = L^{i-k_0} \begin{pmatrix} F_{k_0} \\ G_{k_0} \end{pmatrix} = L^{i-k_0} \begin{pmatrix} \langle \Psi_2, \Phi_1 \rangle \\ \langle \Psi_1, \Phi_2 \rangle \end{pmatrix} = \begin{pmatrix} \langle \Lambda^{2(i-k_0)} \Psi_2, \Phi_1 \rangle \\ \langle \Lambda^{2(i-k_0)} \Psi_1, \Phi_2 \rangle \end{pmatrix} \qquad i \geqslant k_0. \tag{4.11b}$$

By using (2.23), (2.24) and (4.2), we find that

$$\begin{aligned} \tilde{A}_{2i} &= D^{-1}(Q\tilde{F}_i - R\tilde{G}_i) \\ &= D^{-1}[\langle \Lambda^{2(i-k_0)}(\Psi_1^{(1)} - \Lambda\Psi_1), \Phi_1 \rangle - \langle \Lambda^{2(i-k_0)}\Psi_1, \Phi_1^{(-1)} - \Lambda\Phi_1 \rangle] \\ &= \langle \Lambda^{2(i-k_0)}\Psi_1, \Phi_1^{(-1)} \rangle \qquad i \geqslant k_0 \end{aligned} \tag{4.12a}$$

$$\begin{aligned} \tilde{B}_{2i+1} &= (1 - RQ)\tilde{G}_i + Q\tilde{A}_{2i} \\ &= (1 - RQ)\langle \Lambda^{2(i-k_0)}\Psi_1, \Phi_2 \rangle + Q\langle \Lambda^{2(i-k_0)}\Psi_1, R\Phi_2 + \Lambda\Phi_1 \rangle \\ &= \langle \Lambda^{2(i-k_0)+1}\Psi_1, \Phi_2^{(-1)} \rangle \qquad i \geqslant k_0 \end{aligned} \tag{4.12b}$$

$$\tilde{C}_{2i+1}^{(1)} = (1 - RQ)\tilde{F}_i + R\tilde{A}_{2i}^{(1)} = \langle \Lambda^{2(i-k_0)+1}\Psi_2^{(1)}, \Phi_1 \rangle$$

or

$$\tilde{C}_{2i+1} = \langle \Lambda^{2(i-k_0)+1}\Psi_2, \Phi_1^{(-1)} \rangle \qquad i \geqslant k_0. \tag{4.12c}$$

However, it is easy to see from (4.2) that $\tilde{A}_{2i}$, $\tilde{B}_{2i+1}$, $\tilde{C}_{2i+1}$, defined by (4.12), do not satisfy recursion formula (2.9). Therefore, we have to modify $\tilde{A}_{2i}$ so that they satisfy (2.9). In fact, (4.12*a*) implies that we can take

$$\tilde{A}_{2i} = \langle \Lambda^{2(i-k_0)}\Psi_1, \Phi_1^{(-1)} \rangle + \eta_i \qquad i \geqslant k_0 + 1$$

where η_i is a constant; in particular, we take $\eta_i = -\frac{1}{2}h_{2(i-k_0)}$, then we obtain (4.9*c*). Finally, it is easy to verify that due to (4.2) the quantities $\tilde{A}_{2i}$, $\tilde{B}_{2i+1}$, $\tilde{C}_{2i+1}$, defined by (4.9), satisfy (2.9). According to (4.8), we complete the proof. □

For example, for equations (4.4), we obtain from (4.9) that

$$\tilde{A}_0 = A_0 = \tfrac{1}{2} \qquad \tilde{B}_1 = B_1 = Q \qquad \tilde{C}_1 = C_1 = R^{(-1)} \tag{4.13a}$$

$$\tilde{A}_2 = \langle \Psi_1, \Phi_1^{(-1)} \rangle \tag{4.13b}$$

$$\tilde{A}_{2i} = \tfrac{1}{2}(\langle \Lambda^{2i-2}\Psi_1, \Phi_1^{(-1)} \rangle - \langle \Lambda^{2i-2}\Psi_2, \Phi_2^{(-1)} \rangle) \qquad i \geqslant 2 \tag{4.13c}$$

$$\tilde{B}_{2i+1} = \langle \Lambda^{2i-1}\Psi_1, \Phi_2^{(-1)} \rangle \qquad i \geqslant 1 \tag{4.13d}$$

$$\tilde{C}_{2i+1} = \langle \Lambda^{2i-1}\Psi_2, \Phi_1^{(-1)} \rangle \qquad i \geqslant 1. \tag{4.13e}$$

Then, integrals of motion for (4.4) can be found from (4.10). They are

$$\begin{aligned}
F_1 &= \langle \Psi_1, \Phi_1^{(-1)} \rangle + R^{(-1)} Q \\
F_2 &= \langle \Lambda^2\Psi_1, \Phi_1^{(-1)} \rangle - \langle \Lambda^2\Psi_2, \Phi_2^{(-1)} \rangle + \langle \Psi_1, \Phi_1^{(-1)} \rangle^2 \\
&\quad + R^{(-1)}\langle \Lambda\Psi_1, \Phi_2^{(-1)} \rangle + Q\langle \Lambda\Psi_2, \Phi_1^{(-1)} \rangle \\
F_k &= \langle \Lambda^{2k-2}\Psi_1, \Phi_1^{(-1)} \rangle - \langle \Lambda^{2k-2}\Psi_2, \Phi_2^{(-1)} \rangle + \langle \Psi_1, \Phi_1^{(-1)} \rangle(\langle \Lambda^{2k-4}\Psi_1, \Phi_1^{(-1)} \rangle \\
&\quad - \langle \Lambda^{2k-4}\Psi_2, \Phi_2^{(-1)} \rangle) + \sum_{i=2}^{k-2}(\langle \Lambda^{2i-2}\Psi_1, \Phi_1^{(-1)} \rangle - \langle \Lambda^{2i-2}\Psi_2, \Phi_2^{(-1)} \rangle) \\
&\quad \times (\langle \Lambda^{2k-2i-2}\Psi_1, \Phi_1^{(-1)} \rangle - \langle \Lambda^{2k-2i-2}\Psi_2, \Phi_2^{(-1)} \rangle) \\
&\quad + Q\langle \Lambda^{2k-3}\Psi_2, \Phi_1^{(-1)} \rangle + R^{(-1)}\langle \Lambda^{2k-3}\Psi_1, \Phi_2^{(-1)} \rangle \\
&\quad + \sum_{i=1}^{k-2}\langle \Lambda^{2i-1}\Psi_2, \Phi_1^{(-1)} \rangle\langle \Lambda^{2k-2i-3}\Psi_1, \Phi_2^{(-1)} \rangle \qquad k \geqslant 3.
\end{aligned} \tag{4.14}$$

By using the method proposed in [21, 22], we now show how the Lax representation for (4.2) can be deduced from formula (2.5). We find from (4.9)

$$\begin{aligned}
z^{2k_0}\sum_{i=k_0}^{\infty}\tilde{B}_{2i+1}z^{-2i-1} &= \sum_{i=k_0}^{\infty}\langle \Lambda^{2i-2k_0+1}\Psi_1, \Phi_2^{(-1)} \rangle z^{-2i-1+2k_0} \\
&= \sum_{i=0}^{\infty}\sum_{j=1}^{N}\frac{z_j^{2i+1}}{z^{2i+1}}\psi_{1j}\phi_{2j}^{(-1)} \\
&= \sum_{j=1}^{N}\frac{z_j z\psi_{1j}\phi_{2j}^{(-1)}}{z^2 - z_j^2}
\end{aligned} \tag{4.15a}$$

$$z^{2k_0}\sum_{i=k_0}^{\infty}\tilde{C}_{2i+1}z^{-2i-1} = \sum_{j=1}^{N}\frac{z_j z\psi_{2j}\phi_{1j}^{(-1)}}{z^2 - z_j^2} \tag{4.15b}$$

$$z^{2k_0}\sum_{i=k_0+1}^{\infty}\tilde{A}_{2i}z^{-2i} = \frac{1}{2}\sum_{j=1}^{N}\frac{z_j^2(\psi_{1j}\phi_{1j}^{(-1)} - \psi_{2j}\phi_{2j}^{(-1)})}{z^2 - z_j^2}. \tag{4.15c}$$

Let us define

$$\tilde{M}_{k_0} \equiv z^{2k_0}\tilde{\Gamma} = (z^{2k_0}\tilde{\Gamma})_+ + N_0 = V_{k_0} - \Delta + N_0 \tag{4.16}$$

where V_{k_0} is given by (2.13), and

$$N_0 = \sum_{j=1}^{N} \frac{1}{z^2 - z_j^2} \begin{pmatrix} \frac{1}{2} z_j^2 (\psi_{1j}\phi_{1j}^{(-1)} - \psi_{2j}\phi_{2j}^{(-1)}) & z_j z \psi_{1j}\phi_{2j}^{(-1)} \\ z_j z \psi_{2j}\phi_{1j}^{(-1)} & -\frac{1}{2} z_j^2 (\psi_{1j}\phi_{1j}^{(-1)} - \psi_{2j}\phi_{2j}^{(-1)}) \end{pmatrix} \tag{4.17a}$$

$$\Delta = \begin{pmatrix} 0 & 0 \\ 0 & \langle \Psi_1, \Phi_1^{(-1)} \rangle \end{pmatrix}. \tag{4.17b}$$

Since $\tilde{\Gamma}$ satisfies (2.5), $\tilde{M}_{k_0}$ also satisfies (2.5)

$$(E\tilde{M}_{k_0})U - U\tilde{M}_{k_0} = 0.$$

The above formula is valid owing to (4.2). Conversely, it gives the Lax representation for (4.2).

Proposition 4. By substituting expression $\tilde{M}_{k_0}$ for $z^{2k_0}\tilde{\Gamma}$, the stationary zero-curvature equation (2.5) reduces to the Lax representation for (4.2):

$$(E\tilde{M}_{k_0})U - U\tilde{M}_{k_0} = 0 \tag{4.18}$$

with the linear problem equations given by

$$E\psi = U(u, z)\psi \qquad \tilde{M}_{k_0}\psi = \mu\psi. \tag{4.19}$$

Proof. It follows from (2.14), by taking $m = k_0$, that

$$(EV_{k_0})U - UV_{k_0} = \begin{pmatrix} 0 & (1 - RQ)G_{k_0} \\ -(1 - RQ)F_{k_0} & 0 \end{pmatrix}. \tag{4.20a}$$

One finds

$$-(E\Delta)U + U\Delta = \begin{pmatrix} 0 & Q\langle \Psi_1, \Phi_1^{(-1)} \rangle \\ -R\langle \Psi_1^{(1)}, \Phi_1 \rangle & -\frac{1}{z}\langle \Psi_1^{(1)}, \Phi_1 \rangle + \frac{1}{z}\langle \Psi_1, \Phi_1^{(-1)} \rangle \end{pmatrix}. \tag{4.20b}$$

It is easy to calculate the matrix elements of $((EN_0)U - UN_0) \equiv K = (K_{ij})$. For example, we get

$$\begin{aligned} K_{12} &= \sum_{j=1}^{N} \frac{1}{z^2 - z_j^2} [\tfrac{1}{2} z_j^2 Q(\psi_{1j}^{(1)}\phi_{1j} - \psi_{2j}^{(1)}\phi_{2j}) + z_j \psi_{1j}^{(1)}\phi_{2j} - z_j z^2 \psi_{1j}\phi_{2j}^{(-1)} \\ &\quad + \tfrac{1}{2} Q z_j^2 (\psi_{1j}\phi_{1j}^{(-1)} - \psi_{2j}\phi_{2j}^{(-1)})] \\ &= -\langle \Lambda\Psi_1, \Phi_2^{(-1)} \rangle + \sum_{j=1}^{N} \frac{1}{z^2 - z_j^2} [\tfrac{1}{2} z_j^2 Q(\psi_{1j}^{(1)}\phi_{1j} - \psi_{2j}^{(1)}\phi_{2j}) \\ &\quad + z_j \psi_{1j}^{(1)}\phi_{2j} - z_j^3 \psi_{1j}\phi_{2j}^{(-1)} + \tfrac{1}{2} Q z_j^2 (\psi_{1j}\phi_{1j}^{(-1)} - \psi_{2j}\phi_{2j}^{(-1)})]. \end{aligned}$$

Then it is easy to see that the coefficients at $1/(z^2 - z_j^2)$ in (4.18), which are just given by that in K, are satisfied by (4.2*a*) and (4.2*b*) and the remaining terms in K together with (4.20*a*) and (4.20*b*) give rise to (4.2*c*). This completes the proof. □

For example, the Lax representation for equations (4.4) is given by (4.18) with

$$\tilde{M}_1 = \begin{pmatrix} \frac{1}{2}z^2 + \langle \Psi_1, \Phi_1^{(-1)} \rangle & zQ \\ zR^{(-1)} & -\frac{1}{2}z^2 - \langle \Psi_1, \Phi_1^{(-1)} \rangle \end{pmatrix}.$$

The AL hierarchy with sources is defined similarly as for the continuous systems [1,2] as follows

$$E\Psi_1 = \Lambda\Psi_1 + Q\Psi_2 \qquad E\Psi_2 = R\Psi_1 + \Lambda^{-1}\Psi_2 \tag{4.21a}$$

$$E^{(-1)}\Phi_1 = \Lambda\Phi_1 + R\Phi_2 \qquad E^{(-1)}\Phi_2 = Q\Phi_1 + \Lambda^{-1}\Phi_2 \tag{4.21b}$$

$$\begin{pmatrix} Q \\ R \end{pmatrix}_{t_m} = J\frac{\delta H_m}{\delta u} - J\begin{pmatrix} \langle \Psi_2, \Phi_1 \rangle \\ \langle \Psi_1, \Phi_2 \rangle \end{pmatrix}. \tag{4.21c}$$

As a consequence of proposition 4 (note (4.20) for $k_0 = m$), we immediately obtain the following proposition.

Proposition 5. The AL hierarchy with sources (4.21) admits the following discrete zero-curvature representation:

$$U_{t_m} = (E\tilde{M}_m)U - U\tilde{M}_m \tag{4.22}$$

with the linear problem equations given by

$$E\psi = U(u, z)\psi \qquad \psi_{t_m} = \tilde{M}_m\psi \tag{4.23}$$

where

$$\tilde{M}_m = V_m + N_0 - \Delta. \tag{4.24}$$

We can also consider another restricted flow of the AL hierarchy defined as

$$E\Psi_1 = \Lambda\Psi_1 + Q\Psi_2 \qquad E\Psi_2 = R\Psi_1 + \Lambda^{-1}\Psi_2 \tag{4.25a}$$

$$E^{(-1)}\Phi_1 = \Lambda\Phi_1 + R\Phi_2 \qquad E^{(-1)}\Phi_2 = Q\Phi_1 + \Lambda^{-1}\Phi_2 \tag{4.25b}$$

$$\frac{\delta \bar{H}_{k_0}}{\delta Q} = \langle \Psi_2, \Phi_1 \rangle \qquad \frac{\delta \bar{H}_{k_0}}{\delta R} = \langle \Psi_1, \Phi_2 \rangle. \tag{4.25c}$$

It is a Lagrange system. In what follows, we present similar results for (4.25). As before, we define

$$\tilde{\bar{A}}_{2i} = \bar{A}_{2i} \qquad \tilde{\bar{B}}_{2i+1} = \bar{B}_{2i+1} \qquad \tilde{\bar{C}}_{2i+1} = \bar{C}_{2i+1} \qquad i = 0, \ldots, k_0 - 1 \tag{4.26a}$$

$$\tilde{\bar{A}}_{2k_0} = \langle \Psi_2, \Phi_2^{(-1)} \rangle \tag{4.26b}$$

$$\tilde{\bar{A}}_{2i} = -\tfrac{1}{2}(\langle \Lambda^{2k_0-2i}\Psi_1, \Phi_1^{(-1)} \rangle - \langle \Lambda^{2k_0-2i}\Psi_2, \Phi_2^{(-1)} \rangle) \qquad i \geqslant k_0 + 1 \tag{4.26c}$$

$$\tilde{\bar{B}}_{2i+1} = -\langle \Lambda^{2k_0-2i-1}\Psi_1, \Phi_2^{(-1)} \rangle \qquad i \geqslant k_0 \tag{4.26d}$$

$$\tilde{\bar{C}}_{2i+1} = -\langle \Lambda^{2k_0-2i-1}\Psi_2, \Phi_1^{(-1)} \rangle \qquad i \geqslant k_0. \tag{4.26e}$$

Then it is easy to verify that, due to (4.25), the quantities $\tilde{\bar{A}}_{2i}$, $\tilde{\bar{B}}_{2i+1}$, $\tilde{\bar{C}}_{2i+1}$ satisfy (2.29) and, therefore, $\tilde{\bar{\Gamma}}$ satisfies (2.5). According to (4.8), we obtain the following integrals of motion for the Lagrange system (4.25):

$$\bar{F}_k = \sum_{i=0}^{k} \tilde{\bar{A}}_{2i} \tilde{\bar{A}}_{2k-2i} + \sum_{i=0}^{k-1} \tilde{\bar{B}}_{2i+1} \tilde{\bar{C}}_{2k-2i-1} \qquad k = 0, 1, \ldots. \tag{4.27}$$

Let us define

$$\tilde{\bar{M}}_{k_0} \equiv z^{-2k_0} \tilde{\bar{\Gamma}} = \bar{V}_{k_0} + \bar{\Delta} + N_0 \tag{4.28}$$

where $\bar{V}_{k_0}$ is given by (2.31), N_0 is given by (4.17a) and

$$\bar{\Delta} = \frac{1}{2} \begin{pmatrix} \langle \Psi_1, \Phi_1^{(-1)} \rangle + \langle \Psi_2, \Phi_2^{(-1)} \rangle & 0 \\ 0 & -\langle \Psi_1, \Phi_1^{(-1)} \rangle + \langle \Psi_2, \Phi_2^{(-1)} \rangle \end{pmatrix}. \tag{4.29}$$

Then, in a similar way as before, we can show that by substituting expression $\tilde{\bar{M}}_{k_0}$ for $z^{-2k_0} \tilde{\bar{\Gamma}}$, the stationary zero-curvature equation (2.5) reduces to the Lax representation for (4.25):

$$(E \tilde{\bar{M}}_{k_0}) U - U \tilde{\bar{M}}_{k_0} = 0 \tag{4.30}$$

with the linear problem equations given by

$$E\psi = U(u, z)\psi \qquad \tilde{\bar{M}}_{k_0} \psi = \mu \psi. \tag{4.31}$$

Also, the following AL hierarchy with sources:

$$E\Psi_1 = \Lambda \Psi_1 + Q \Psi_2 \qquad E\Psi_2 = R\Psi_1 + \Lambda^{-1} \Psi_2 \tag{4.32a}$$

$$E^{(-1)} \Phi_1 = \Lambda \Phi_1 + R \Phi_2 \qquad E^{(-1)} \Phi_2 = Q \Phi_1 + \Lambda^{-1} \Phi_2 \tag{4.32b}$$

$$\begin{pmatrix} Q \\ R \end{pmatrix}_{t_m} = J \frac{\delta \bar{H}_m}{\delta u} - J \begin{pmatrix} \langle \Psi_2, \Phi_1 \rangle \\ \langle \Psi_1, \Phi_2 \rangle \end{pmatrix} \tag{4.32c}$$

admits the following discrete zero-curvature representation:

$$U_{t_m} = (E \tilde{\bar{M}}_m) U - U \tilde{\bar{M}}_m \tag{4.33}$$

with the linear problem equations given by

$$E\psi = U(u, z)\psi \qquad \psi_{t_m} = \tilde{\bar{M}}_m \psi \tag{4.34}$$

where

$$\tilde{\bar{M}}_m = \bar{V}_m + N_0 + \bar{\Delta}.$$

Finally, we consider the continuous limit of the following AL hierarchy with sources:

$$E\Psi_1 = \Lambda\Psi_1 + Q\Psi_2 \qquad E\Psi_2 = R\Psi_1 + \Lambda^{-1}\Psi_2 \tag{4.35a}$$

$$E^{(-1)}\Phi_1 = \Lambda\Phi_1 + R\Phi_2 \qquad E^{(-1)}\Phi_2 = Q\Phi_1 + \Lambda^{-1}\Phi_2 \tag{4.35b}$$

$$\begin{pmatrix} Q \\ R \end{pmatrix}_{t_m} = \frac{\rho}{(-2)^m m h^{m+1}} J\left\{ \sum_{i=0}^{m} C_m^i (-1)^{m-i}\left[\begin{pmatrix} F_i \\ G_i \end{pmatrix} + (-1)^{m-1} \begin{pmatrix} \bar{F}_i \\ \bar{G}_i \end{pmatrix}\right] - \begin{pmatrix} \langle \Psi_2, \Phi_1\rangle \\ \langle \Psi_1, \Phi_2\rangle \end{pmatrix} \right\} \tag{4.35c}$$

and the AKNS hierarchy with sources [1, 2]

$$\tilde{\Psi}_{1x} = -\tilde{\Lambda}\tilde{\Psi}_1 + q\tilde{\Psi}_2 \qquad \tilde{\Psi}_{2x} = r\tilde{\Psi}_1 + \tilde{\Lambda}\tilde{\Psi}_2 \tag{4.36a}$$

$$\tilde{\Phi}_{1x} = \tilde{\Lambda}\tilde{\Phi}_1 - r\tilde{\Phi}_2 \qquad \tilde{\Phi}_{2x} = -q\tilde{\Phi}_1 - \tilde{\Lambda}\tilde{\Phi}_2 \tag{4.36b}$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_m} = \rho J_0\left[\begin{pmatrix} c_{m+2} \\ b_{m+2} \end{pmatrix} - \begin{pmatrix} \langle \tilde{\Psi}_2, \tilde{\Phi}_1\rangle \\ \langle \tilde{\Psi}_1, \tilde{\Phi}_2\rangle \end{pmatrix} \right] \tag{4.36c}$$

where

$$\tilde{\Psi}_i = (\tilde{\psi}_{i1}, \ldots, \tilde{\psi}_{iN})^t \qquad \tilde{\Phi}_i = (\tilde{\phi}_{i1}, \ldots, \tilde{\phi}_{iN})^t \qquad i = 1, 2$$

$$\tilde{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N).$$

Owing to (3.15) and (3.24), it is easy to find following proposition.

Proposition 6. Under transformation (3.9) with

$$\beta = [(-2)^{m+1} m h^{m+2}]^{1/2}$$

the AL hierarchy with sources (4.35) goes to the AKNS hierarchy with sources (4.36) in the continuous limit.

As a consequence, all restricted flows of AL hierarchy (3.25), which are defined as the stationary equations of (4.35), go to the restricted flows of AKNS hierarchy (3.2), which are defined as the stationary equations of (4.36), in the continuous limit. This continuous limit may be used for deriving numerical schemes for calculation of trajectories of restricted flows of the AKNS hierarchy.

For example, the restricted flow of AKNS hierarchy for $m = 1$ reads

$$\tilde{\Psi}_{1x} = -\tilde{\Lambda}\tilde{\Psi}_1 + q\tilde{\Psi}_2 \qquad \tilde{\Psi}_{2x} = r\tilde{\Psi}_1 + \tilde{\Lambda}\tilde{\Psi}_2 \tag{4.37a}$$

$$\tilde{\Phi}_{1x} = \tilde{\Lambda}\tilde{\Phi}_1 - r\tilde{\Phi}_2 \qquad \tilde{\Phi}_{2x} = -q\tilde{\Phi}_1 - \tilde{\Lambda}\tilde{\Phi}_2 \tag{4.37b}$$

$$\tfrac{1}{4}(r_{xx} - 2qr^2) = \langle \tilde{\Psi}_2, \tilde{\Phi}_1\rangle \qquad \tfrac{1}{4}(q_{xx} - 2q^2 r) = \langle \tilde{\Psi}_1, \tilde{\Phi}_2\rangle. \tag{4.37c}$$

Define

$$q_1 = q \qquad q_2 = r \qquad p_1 = -\tfrac{1}{4} r_x \qquad p_2 = -\tfrac{1}{4} q_x$$

then (4.37) can be written as a Hamiltonian system

$$\tilde{\Psi}_{ix} = \frac{\partial \tilde{H}_1}{\partial \tilde{\Phi}_i} \qquad \tilde{\Phi}_{ix} = -\frac{\partial \tilde{H}_1}{\partial \tilde{\Psi}_i} \qquad q_{ix} = \frac{\partial \tilde{H}_1}{\partial p_i} \qquad p_{ix} = -\frac{\partial \tilde{H}_1}{\partial q_i} \qquad i = 1, 2 \tag{4.38}$$

with

$$\tilde{H}_1 = -\langle \tilde{\Lambda}\tilde{\Psi}_1, \tilde{\Phi}_1 \rangle + \langle \tilde{\Lambda}\tilde{\Psi}_2, \tilde{\Phi}_2 \rangle + q_1 \langle \tilde{\Psi}_2, \tilde{\Phi}_1 \rangle + q_2 \langle \tilde{\Psi}_1, \tilde{\Phi}_2 \rangle - 4p_1 p_2 + \tfrac{1}{4} q_1^2 q_2^2.$$

It was shown in [1, 2] that the finite-dimensional Hamiltonian system (4.38) is completely integrable in the Liouville sense.

The restricted flow of AL hierarchy for $m = 1$ is

$$E\Psi_1 = \Lambda\Psi_1 + Q\Psi_2 \qquad E\Psi_2 = R\Psi_1 + \Lambda^{-1}\Psi_2 \tag{4.39a}$$

$$E^{(-1)}\Phi_1 = \Lambda\Phi_1 + R\Phi_2 \qquad E^{(-1)}\Phi_2 = Q\Phi_1 + \Lambda^{-1}\Phi_2 \tag{4.39b}$$

$$R^{(1)} + R^{(-1)} - RR^{(-1)}Q - RR^{(1)}Q - 2R = \langle \Psi_2, \Phi_1 \rangle \tag{4.39c}$$

$$Q^{(1)} + Q^{(-1)} - QQ^{(-1)}R - QQ^{(1)}R - 2Q = \langle \Psi_1, \Phi_2 \rangle. \tag{4.39d}$$

Proposition 6 implies that under transformation (3.9) with $\beta = 2h^{3/2}$, restricted flow (4.39) is the discrete version of restricted flow (4.37).

Acknowledgment

One of us (YBZ) thanks the Swedish Institute for financial support.

References

[1] Ragnisco O and Rauch-Wojciechowski S 1992 *Inverse Problems* **8** 245

[2] Zeng Yunbo and Li Yishen The general scheme for higher-order decompositions of zero-curvature equations associated with $sl(2)$ *Preprint*

[3] Ragnisco O 1992 *Phys. Lett.* **167A** 165

[4] Ragnisco O and Rauch-Wojciechowski S Integrable maps for the Garnier and the Neumann system *Preprint*

[5] Zeng Yunbo, Rauch-Wojciechowski S and Li Yishen Factorization of flows of the Toda hierarchy and new integrable symplectic maps *Preprint*

[6] Zeng Yunbo and Rauch-Wojciechowski S Continuous limit for the Kac–Van Moer-beke hierarchy and for their restricted flows *Preprint*

[7] Herbst B M and Ablowitz M J 1993 *Comput. Phys.* **105** 122

[8] Yoshida H 1993 *Celest. Mech. Dynam. Astron.* **56** 27

[9] Rauch-Wojciechowski S 1992 *Phys. Lett.* **170A** 91

[10] Zeng Yunbo 1991 *Phys. Lett.* **160A** 541

[11] Rauch-Wojciechowski S, Jiang Zhuhan and Bullough R K 1986 *Phys. Lett.* **117A** 399

[12] Watson K M, West B J and Cohen B I 1978 *J. Fluid Mech.* **77** 185

[13] Meiss J D 1979 *Phys. Rev.* A **19** 1780

[14] Moser J and Veselov A P 1991 *Commun. Math. Phys.* **139** 217

[15] Ablowitz M J and Ladik J 1975 *J. Math. Phys.* **16** 598; 1976 **17** 1011

[16] Tu Guizhang 1990 *J. Phys. A: Math. Gen.* **23** 3903

[17] Gerdjikov V S, Ivanov M I and Kulish P P 1984 *J. Math. Phys.* **25** 25

[18] Schilling R J 1989 *J. Math. Phys.* **30** 1487

[19] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 249

[20] Bruschi M, Ragnisco O, Santini P M and Tu Guizhang 1991 *Physica* **49D** 273

[21] Antonowicz M and Rauch-Wojciechowski S 1993 *Inverse Problems* **9** 201

[22] Zeng Yunbo and Li Yishen 1993 *J. Phys. A: Math. Gen.* **26** L273